Math 409 Midterm 2 practice #1

Name: ______________________

This exam has 4 questions, for a total of 100 points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (40 pts)

In each of the following eight cases, indicate whether the given statement is true or false. No justification is necessary.

(a) Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences in $\mathbb{R}$. Then the sequence $\{x_n y_n\}$ converges.

Solution: True.

(b) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ does not exist.

Solution: False.

(c) Let $\{x_n\}$ be a sequence such that $x_n \in (0, 1)$ for every $n \in \mathbb{N}$. Then $\{x_n\}$ has a subsequence which is Cauchy.

Solution: True. (This essentially follows from Bolzano-Weierstrass theorem.)

(d) $\lim_{x\to\infty} \dfrac{x - 2x^2 + 5x^3}{6 - x + x^2} = \infty$

Solution: True.

(e) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ with the property that each of its subsequences has a convergent subsequence. Then $\{x_n\}$ is bounded.

Solution: True. (One can prove this by contradiction.)

(f) If a function is differentiable on $\mathbb{R}$, then it is uniformly continuous on $\mathbb{R}$.

Solution: False.

(g) Let f be a function which is uniformly continuous on $\mathbb{R}$. Then the function g defined by $g(x) = f(f(x))$ for all $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Solution: True.

(h) If $f\colon (0, 1) \to \mathbb{R}$ is continuous and bounded, then f is uniformly continuous.

Solution: False.

Question 2. (20 pts)

(a) Let f be a function defined on an open interval containing a given point a. State what it means for $f(x)$ to converge to a number L as x approaches a.

Solution: Omitted. You can find it in the textbook.

(b) Let $a \in \mathbb{R}$ and let f and g be functions on $\mathbb{R}$ such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Prove directly from the definition of a limit that $\lim_{x\to a}(f+g)(x)$ exists.

Solution: By assumption, for $\forall\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(x) - f(a)| < \varepsilon/2$$

for all x with $0 < |x - a| < \delta_1$; similarly, there exists $\delta_2 > 0$ such that

$$|g(x) - g(a)| < \varepsilon/2$$

for all x with $0 < |x - a| < \delta_2$.
Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|(f + g)(x) - (f + g)(a)| < \varepsilon$$

for all x with $0 < |x - a| < \delta$.

Question 3. (20 pts)

(a) State the Extreme Value Theorem.

Solution: Omitted. You can find it in the textbook.

(b) Give an example of a function f which is bounded on $[0,1]$ but does not have a maximum on $[0,1]$.

Solution: Define

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & x = 1 \end{cases}$$

Question 4. (20 pts)

(a) State the Intermediate Value Theorem.

Solution: Omitted. You can find it in the textbook.

(b) Assuming the fact that the function $\cos x$ is continuous on $\mathbb{R}$, prove that there exists an $x \in \mathbb{R}$ such that $x^6 + x^4 + 1 = 2\cos x^3$.

Solution: Consider the function $g(x) = x^6 + x^4 + 1 - 2\cos x^3$. On the interval $[0, 1]$, we have

$$g(0) = -1 \text{ and } g(1) = 3 - 2\cos(1) > 0.$$

So we have $g(0) \leq 0 \leq g(1)$. It follows from the intermediate value theorem that there exists $x_0 \in [0, 1]$ such that $g(x_0) = 0$, that is, $x_0^6 + x_0^4 + 1 = 2\cos x_0^3$.